CONCERNING CERTAIN EQUICONTINUOUS SYSTEMS OF CURVES*

BY

ROBERT L. MOORE

In order that a system G of open curves lying in a given plane S should be equivalent, from the standpoint of analysis situs,† to a complete† system of parallel lines in S it is not sufficient that through each point of S there should

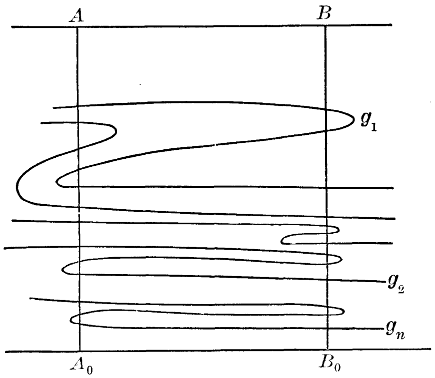

FIG. 1.

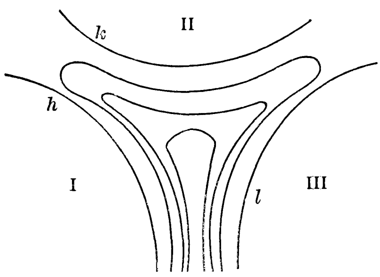

FIG. 2.

pass one and only one curve of the system G. Consider the examples indicated in Figs. 1 and 2.‡ In each of these examples through each point of the plane there is one and only one curve of the system in question but the system

* Cf. papers presented to the Society, April 28 and October 27, 1917.

† A *complete system* of parallel lines in a plane S is the set of all lines in S parallel to a given line. A system G of open curves is said to be equivalent, from the standpoint of analysis situs, to such a system of lines L if there is a one to one continuous transformation of S into itself which carries G into L.

‡ In the case roughly indicated by Fig. 1, $A_0 B_0$ and AB are two parallel lines at a distance apart equal to 1. These lines both belong to the system G and so does every line which is parallel to them but which does not lie between them. For each positive integer n, g_n is an open curve belonging to G such that (1) there is an interval of g_n that contains a point of $B_0 B$ but has its endpoints on $A_0 A$, (2) every point of g_n is at a distance of less than $1/n$ from the line $A_0 B_0$. Of course, as is indicated in Fig. 1 for the case $n = 1$, there does not exist, on *every* curve of G that lies between g_n and g_{n+1}, an interval that contains a point of $B_0 B$ and has its endpoints on $A_0 A$.

In the example indicated in Fig. 2, the open curves h, k and l belong to G. To obtain the curves of G which lie in domain I, II or III construct through each point of that domain an open curve parallel and congruent to h, k or l respectively. Each curve of G that lies in the domain bounded by h, k and l lies as is roughly suggested in the figure.

is not in one to one continuous correspondence with a complete system of parallel lines. Let G_1 and G_2 be the system of curves represented in Figs. 1 and 2 respectively. The system G_1 is not equicontinuous.* That is to say it is not true that for every positive number ϵ there exists a positive number δ_ϵ such that if P_1 and P_2 are points on some curve g of G at a distance apart less than δ_ϵ then that arc of g which has P_1 and P_2 as its endpoints lies wholly within some circle of radius ϵ. The system G_2 is equicontinuous but fails to be what I will call inversely equicontinuous.

Definition 1. A system of curves G is *equicontinuous with respect to a given point-set* M if for every positive number ϵ there exists a positive number $\delta_{M\epsilon}$ such that if P_1 and P_2 are two points of M at a distance apart less than $\delta_{M\epsilon}$ and lying on a curve g of the system G then that arc of g which has P_1 and P_2 as endpoints lies wholly within some circle of radius ϵ.

Definition 2. A system of curves G is *inversely equicontinuous with respect to a point-set* M if for every positive number ϵ there exists a positive number $\delta_{M\epsilon}$ such that if P_1 and P_2 are two points of M at a distance apart less than ϵ and lying on a curve g of the system G then that interval of g which has P_1 and P_2 as endpoints lies wholly within a circle of radius $\delta_{M\epsilon}$.

I will show that if G is a system of open curves lying in S such that through each point of S there is just one curve of G, then in order that the system G should be equivalent, from the standpoint of analysis situs, to a complete system of parallel straight lines it is necessary and sufficient that it should be both equicontinuous and inversely equicontinuous with respect to every bounded set of points. Additional theorems of a related nature will also be established.

Theorem 1. *Suppose that, in a given plane* S, $ABCD$ *is a rectangle and* G *is a set of arcs such that* (1) *through each point of the point-set* $\overline{R}$, *composed of* $ABCD$ *and its interior* R, *there is just one arc of* G, (2) BC *and* AD *are arcs of* G, (3) *every arc of* G (*with the exception of* BC *and* AD) *lies entirely within* $ABCD$ *except that its endpoints are on* AB *and* CD *respectively,* (4) *the set of arcs* G *is equicontinuous.*

Then there is a one to one continuous transformation of the plane S *into itself which transforms the rectangle* $ABCD$ *into a rectangle* $A'\,B'\,C'\,D'$ *and transforms the set of arcs* G *into the set of all straight line intervals which are parallel to* $A'\,D'$ *and lie between* $A'\,D'$ *and* $B'\,C'$ (*except that one of them coincides with* $A'\,D'$ *and another with* $B'\,C'$) *and are terminated by* $A'\,B'$ *and* $C'\,D'$.

The truth of this theorem will be established with the help of a lemma. This lemma will be proved first.

Definition 3. A connected domain K is said to be a *simple domain with*

* Cf. G. Ascoli, *Sulle curve limiti di una varietà data di curve,* Memorie della Reale Accademia dei Lincei, vol. 18 (1884), pp. 521–586.

respect to a set of arcs G satisfying the conditions stated in the hypothesis of Theorem 1 if (1) every point of K is within $ABCD$, (2) K contains the whole of every G-interval* whose endpoints are in K, (3) there exist two G-arcs g_1 and g_2 such that (a) g_1 lies above g_2, every point of K is between g_1 and g_2 and both g_1 and g_2 have points in common with the boundary of K, (b) the set of all those points that the boundary of K has in common with g_i is an interval t_i of g_i ($i = 1, 2$), (c) no point of t_1 or of t_2 is a limit point of a point-set which lies between g_1 and g_2 and contains no point of K. The interval t_i minus its endpoints will be called the upper base, and the interval t_2 minus its endpoints will be called the lower base, of the domain K.

LEMMA 1. *If G is a set of arcs satisfying the conditions stated in the hypothesis of Theorem* 1 *and K is a simple domain with respect to G, then any point on the upper base of K can be joined to any point on its lower base by a simple continuous arc that lies wholly in K and does not have more than one point in common with any arc of the set G.*

Proof. If P is a point of R and ϵ is a positive number let $R_{P\epsilon}$ denote the set of all points X such that X lies on a G-interval whose endpoints are both within a circle of radius ϵ with center at P. If for a given point P and a given pair of positive numbers e and ϵ, such that $e \leqq \epsilon$, the point-set R_{Pe} has points between two distinct G-arcs g_1 and g_2 and also has points on g_1 and points on g_2, the set of all those points of R_{Pe} that lie between g_1 and g_2 will be called an elemental region of rank ϵ.† It may be easily proved that if ϵ is a positive number each point of K is in some elemental region of rank ϵ which lies together with its boundary wholly in the point-set K^* composed of K and its two bases. Such an elemental region will be called a K-element of rank ϵ. If E and F are two points of K^* and E is above F, a *chain of K-elements from E to F* or *from F to E* or *joining E to F* or *F to E* is a finite set of K-elements $K_1, K_2, K_3, \cdots, K_n$ such that (1) E belongs to the upper base of K_1 and F belongs to the lower base of K_n, (2) for each i ($1 \leqq i \leqq n$) the lower base of K_i and the upper base of K_{i+1} lie on the same arc of the set G and have points in common and the set of all their common points is a segment t_i. The point-set $K_1 + K_2 + K_3 + \cdots + K_n + t_1 + t_2 + t_3 + \cdots + t_{n-1}$ is a simple domain. It will be called the domain associated with the chain $K_1, K_2, \cdots, K_n$. Suppose that E is a point on the upper base of K, F is a point on its lower base and ϵ is a positive number. I will show that E can be joined to F by a chain of K-elements of rank ϵ. Let $\overline{K}$ denote the set of all those points of K

* If G is a set of arcs or curves a *G-arc* or a *G-curve* is an arc or a curve of the set G. A *G-interval* is an interval (and a G-segment is a segment) of such an arc or curve. If G is a set satisfying the conditions stated in the hypothesis of Theorem 1, the G-arc g_1 is said to be *above* the G-arc g_2 if it lies between g_2 and BC. If P is a point of $\overline{R}$, g_P denotes that arc of G which contains P. If P_1 and P_2 are points of $\overline{R}$, P_1 will be said to lie above P_2 in case g_{P_1} is above g_{P_2}.

† According to this definition if $\epsilon_1 < \epsilon_2$ every elemental region of rank ϵ_1 is also of rank ϵ_2.

that lie on arcs of G below the arc g_E and that can be joined to E by chains of K-elements of rank ϵ. There exists a K-element of rank ϵ whose upper base contains the point E and every such K-element contains points in common with some g-arc lying below g_E. It follows that the set $\overline{K}$ exists.

Suppose that WZ is an arc of G that contains a point of $\overline{K}$. The set of points common to WZ and K is a segment $W'Z'$. Every point of $W'Z'$ must belong to $\overline{K}$. For suppose this is not the case. Then the segment $W'Z'$ is the sum of two mutually exclusive point-sets S_1 and S_2 such that S_1 is a subset of $\overline{K}$ but no point of S_2 belongs to $\overline{K}$. There exists a point P which either belongs to S_1 and is a limit point of S_2 or belongs to S_2 and is a limit point of S_1. In the first case there is a chain α_2 of K-elements of rank ϵ from E to P. The lower base of the last element of this chain is a segment of $W'Z'$ containing P. Since P is a limit point of S_2 this segment must contain at least one point P_2 of S_2. Thus α_2 is a chain of K-elements of rank ϵ from E to P_2. Thus the supposition that S_1 contains a limit point of S_2 leads to a contradiction. Suppose now that S_2 contains a point P which is a limit point of S_1. There exists (Fig. 3) a K-element e of rank ϵ whose lower base

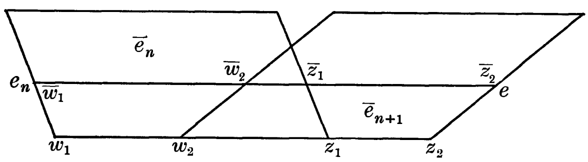

FIG. 3.

W_2Z_2 is a segment of $W'Z'$ containing P. Since P is a limit point of S_1 there exists on the segment W_2Z_2 a point P_1 belonging to S_1. There exists a chain $e_1, e_2, e_3, \cdots, e_n$ of K-elements of rank ϵ from E to P_1. The lower base of the last element e_n of this chain is a segment W_1Z_1 containing P_1. There exist a G-arc $\bar{g}$ and two segments $\overline{W}_1\overline{Z}_1$ and $\overline{W}_2\overline{Z}_2$ such that (1) $\overline{W}_1\overline{Z}_1$ is the set of all points common to e_n and $\bar{g}$, (2) $\overline{W}_2\overline{Z}_2$ is the set of all points common to e and $\bar{g}$, (3) $\overline{W}_1\overline{Z}_1$ and $\overline{W}_2\overline{Z}_2$ have a segment in common. Let $\bar{e}_n$ denote that part of e_n which lies between $\bar{g}$ and the arc of G that contains the upper base of e_n. Let $\bar{e}_{n+1}$ denote that part of e which lies between $\bar{g}$ and W_2Z_2. The set of elements $e_1, e_2, e_3, \cdots, e_{n-1}, \bar{e}_n, \bar{e}_{n+1}$ is a chain of K-elements of rank ϵ from E to P. It is thus established that if one point of $W'Z'$ belongs to $\overline{K}$ then so does every other point of $W'Z'$. It has been shown that if a G-arc above g_F contains a point of $\overline{K}$ then so must some lower arc of G. It follows that if F does not belong to $\overline{K}$ there exists an arc XY which is the uppermost arc of G that contains no point of $\overline{K}$. Let P denote a point of K on the arc XY. There exists a K-element e of rank ϵ whose lower base contains P. The set G contains an arc g that intersects e in a segment MN.

Let $\bar{P}$ denote a point of MN. There exists a chain of K-elements of rank ϵ from E to $\bar{P}$. If to this chain of elements there is added that portion of the K-element e which lies between g and XY there is obtained a chain of K-elements of rank ϵ from E to P. Thus the supposition that E can not be joined to F by a chain of K-elements of rank ϵ leads to a contradiction. It follows that there exists a simple chain $e_{11}, e_{12}, e_{13}, \cdots, E_{1n}$ of K-elements of rank 1 from E to F. Let K_1 denote the domain associated with this chain. There exists a simple chain of K_1-elements of rank 1 from E to F. This process may be continued. It follows that there exists a sequence of simple chains $C_1, C_2, C_3, \cdots$ from E to F such that if, for each n, K_n denotes the domain associated with C_n then (1) every link of C_{n+1} is a K_n-element of rank $1/n$, (2) K'_{n+1} is a subset of the point-set composed of K_n plus its bases. Let t denote the set of all points $[X]$ such that X belongs to every K_n. With the aid of the fact that the set G is equicontinuous, it can be proved* that t is a simple continuous arc from E to F and that it does not have more than one point in common with any given arc of the set G. The truth of Lemma 1 is thus established.

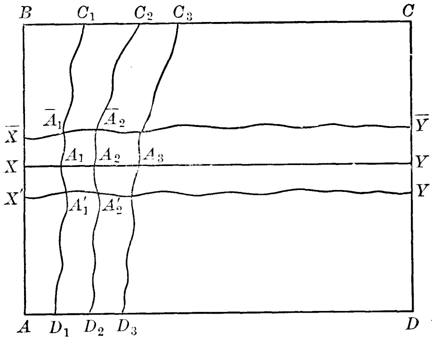

FIG. 4.

Proof of Theorem 1. If X is a point of AB and XY is that arc of G which has X as one of its endpoints, it may be easily proved with the aid of the Heine-Borel Theorem that there exists on XY a finite set of points $A_1, A_2, A_3, \cdots, A_n$ in the order $XA_1 A_2 A_3 A_4 \cdots A_{n-1} A_n Y$ such that each of the intervals $XA_1, A_1 A_2, \cdots, A_{n-1} A_n, A_n Y$ of the arc XY lies wholly within some circle of radius 1. Let $C_1, C_2, C_3, \cdots, C_n$ denote n points in the order $BC_1 C_2 C_3 \cdots C_{n-1} C_n C$ on the arc BC and let $D_1, D_2, D_3, \cdots, D_n$ denote n points in the order $AD_1 D_2 \cdots D_{n-1} D_n D$ on the arc AD. With the use of Lemma 1 it is easily established that there exist (Fig. 4) two sets of arcs $A_1 C_1, A_2 C_2, A_3 C_3, \cdots, A_n C_n$ and $A_1 D_1, A_2 D_2, A_3 D_3, \cdots, A_n D_n$ such that no arc of either set has a point in common with any other arc of that set and such that, for every n, (1) $A_n C_n$ lies except for its endpoints entirely within $ABCD$ and between XY and BC, (2) $A_n D_n$ lies, except for its endpoints, entirely within $ABCD$ and between XY and AD, (3) neither $A_n C_n$ nor $A_n D_n$ has more than one point in common with any one arc of the set G. It is easy to show that there exist two points X' and $\bar{X}$ in the order $AX' X\bar{X}B$

* Cf. the proof of Theorem 15 of my paper *On the foundations of plane analysis situs*, these Transactions, vol. 17 (1916), pp. 136–139.

and two arcs $X' Y'$ and $\bar{X}\bar{Y}$ belonging to G such that if for every i $(1 \leqq i \leqq n)$ A'_i is the point in which $X' Y'$ intersects $A_i D_i$ and $\bar{A}_i$ is the point in which $\bar{X}\bar{Y}$ intersects $A_i C_i$ then the closed curve bounded by the intervals $A'_i A_i$, $A_i \bar{A}_i$, $\bar{A}_i \bar{A}_{i+1}$, $\bar{A}_{i+1} A_{i+1}$, $A_{i+1} A'_{i+1}$ and $A'_{i+1} A'_i$ of the arcs $A_i D_i$, $A_i C_i$, $\bar{X}\bar{Y}$, $A_{i+1} C_{i+1}$, $A_{i+1} D_{i+1}$ and $X' Y'$ respectively (Fig. 4) lies entirely within some circle of radius 1. For each point X of AB make a similar construction and apply the Heine-Borel Theorem to the set of segments $[X' \bar{X}]$. If certain arcs are properly continued there will result a double ruling* T_1 of $ABCD$ such that (1) the arcs of one of its single rulings are arcs of G and each arc of its other single ruling has its endpoints on BC and AD respectively and has just one point in common with each arc of the set G, (2) each of the subdivisions into which T_1 divides $ABCD$ lies within some circle of radius 1. In a similar way each subdivision α of this set can itself be subdivided by a double ruling $T_{1\alpha}$ such that (1) each arc of one of its single rulings is an interval of an arc of G, (2) each arc of its other single ruling has its endpoints on the arcs which form respectively the upper and the lower base of α and no arc of this ruling has more than one point in common with any arc of G, (3) each of the subdivisions into which $T_{1\alpha}$ divides α is within a circle of radius $1/2$. It follows that there exists a double ruling T_2 satisfying the Conditions (1) and (2) stated above as being satisfied by T_1 and also satisfying the additional condition that each of its subdivisions is within some circle of radius $1/2$, for every α each arc of $T_{1\alpha}$ being an interval of an arc of one or the other of the rulings of T_2. This may be continued. It follows that there exists an infinite sequence of double rulings $T_1, T_2, T_3, \cdots$ such that for every n, (1) T_n satisfies the conditions (1) and (2) stated above for T_1, (2) each arc of T_n is an arc of T_{n+1}, (3) each subdivision of T_n is within a circle of radius $1/n$. Let β be the set of all arcs $[t]$ such that, for some n, t belongs to one of the rulings of T_n and has its endpoints on AD and BC respectively. If P is a point on BC which is not an endpoint of an arc of the set β then there exists just one arc t_P that has one endpoint at P and the other on AD, lies except for its endpoints entirely within $ABCD$ and has no point in common with any arc of the set β. Let γ be the set of all such arcs t_P for all such points P. Let G' denote the set of arcs composed of all the arcs of β together with all the arcs of γ and the straight intervals AB and CD. If P is a point on or within the rectangle $ABCD$ let h_P denote the distance from A to the point of intersection of AD with that arc of G' that passes through P. Let k_P denote the distance from A to the point in which AB intersects that arc of G which passes through P. Let AD be the axis of X and AB the axis of Y in a rectangular system of coördinates. If P is on or within the rectangle $ABCD$ let

* Cf. my paper *Concerning a set of postulates for plane analysis situs*, these Transactions, vol. 20 (1919), p. 172 (footnote) and pp. 172–175.

P' denote the point whose coördinates are (h_P, k_P). Let $\overline{T}$ denote the transformation of $\overline{R}$ into itself such that if P is any point of $\overline{R}$ then $\overline{T}(P) = P'$. It is easy to see that the transformation $\overline{T}$ is continuous and that there exists a continuous transformation T, of S into itself, which reduces to $\overline{T}$ on $\overline{R}$. The transformation T satisfies all the requirements of Theorem 1.

THEOREM 2. *If, in a plane S, G is a set of open curves such that through each point of S there is just one curve of G, then in order that the set of curves G should be in one to one continuous correspondence with a complete system of parallel lines in S it is necessary and sufficient that the set G should be both equicontinuous and inversely equicontinuous with respect to every bounded set of points.*

That this condition is necessary may be easily seen. I will show that it is sufficient.

Proof. Suppose that G is a set of open curves such that (1) through each point of S there is just one curve of G, (2) G is both equicontinuous and inversely equicontinuous with respect to every bounded set of points. I will first show that of any three distinct curves of the set G one separates the other two from each other.

Suppose on the contrary that there exist three open curves h, k and l of the set G such that no one of them separates the other two. Then the set of all points $[P]$ such that P is between every two of the curves h, k and l is a domain D. Every curve of G which contains a point of D lies wholly in D. If g is any curve of G lying wholly in D then either (1) g separates one of the curves h, k and l from the other two or (2) two of the curves h, k and l are such that if they be designated as $\overline{h}$ and $\overline{k}$ respectively and the third one be designated as $\overline{l}$ then there exists a ray AB of $\overline{h}$, a ray CD of $\overline{k}$ and an arc AC lying except for its endpoints wholly in D such that the rays AB and CD and the arc AC constitute the common boundary of a domain E which contains g and is a subset of D. A curve g satisfying condition (1) will be called a curve of class I with respect to that one of the curves h, k and l which it separates from the other two, and a curve satisfying condition (2) will be called a curve of class II with respect to $\overline{h}$ and $\overline{k}$. Suppose there exist curves of class I with respect to h. It is clear that of every two such curves one of them separates the other one from h and is separated by the other one from k and from l. I will show that there is a last curve of class I with respect to h, that is to say there is one that separates every other one from k and from l. Suppose this is not the case. Let K denote a point of k and L a point of l and let KL denote an arc which lies except for its endpoints entirely in D. In view of the fact that the set of curves G is inversely equicontinuous with respect to the bounded point-set KL, it is clear that there exist two other points K' and L' on k and l respectively and an arc $K'L'$ lying, except for its endpoints,

wholly in D and having no point in common with KL such that (1) the rays $K'K$ and $L'L$ of k and l respectively, together with the arc $K'L'$ and the curve h, constitute the complete boundary of a domain α which is a subset of D and (2) no arc of G with endpoints on KL contains a point of $K'L'$. There must exist a point-set β (Fig. 5) which is a subset of $\alpha +$ ray $K'K$

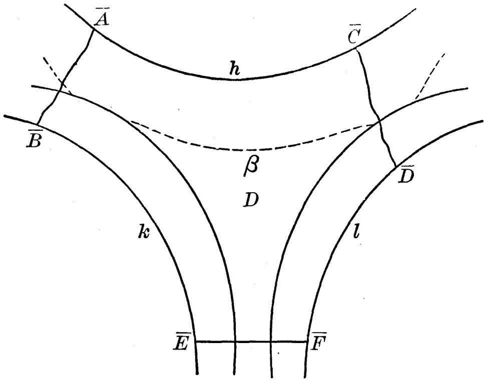

FIG. 5.

$+$ ray $L'L$ such that β is the complete boundary of the set of all points $[X]$ such that X is separated from k and from l by some curve of class I with respect to h. The point-set β is connected and contains points in α. For each such point P there exists through P a curve g_P of the set G. Let $\bar{g}_P$ denote the set of all those points that are common to g_P and β. The point-set $\bar{g}_P$ is a closed proper subset of the connected point-set β and has no point in common with k or with l. It follows that if $\bar{P}$ is a definite point of β lying in α the point-set $\bar{g}_{\bar{P}}$ contains a point P_0 which is the sequential limit point of a sequence of points $P_1, P_2, P_3, \cdots$, all belonging to β and lying in α such that (1) no two of the point-sets $\bar{g}_{P_1}, \bar{g}_{P_2}, \bar{g}_{P_3}, \cdots$ lie on the same curve of the set G and (2) the point-set $P_1 + P_2 + P_3 + \cdots$ is within some closed curve J that lies wholly in D. There exist six distinct points $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$, $\bar{E}$ and $\bar{F}$ and arcs $\bar{A}\bar{B}$, $\bar{C}\bar{D}$, $\bar{E}\bar{F}$ such that (1) $\bar{A}$ and $\bar{C}$ are on h, $\bar{B}$ and $\bar{E}$ are on k, and $\bar{D}$ and $\bar{F}$ are on l, (2) each of the arcs $\bar{A}\bar{B}$, $\bar{C}\bar{D}$ and $\bar{E}\bar{F}$ lies, except for its endpoints, in the domain D and no two of them have a point in common, (3) the curve J is wholly within the closed curve $\bar{J}$ formed by the arcs $\bar{A}\bar{B}$, $\bar{E}\bar{F}$ and $\bar{C}\bar{D}$ together with the intervals $\bar{A}\bar{C}$, $\bar{B}\bar{E}$ and $\bar{D}\bar{F}$ of the curves h, k and l respectively. There does not exist more than one integer n such that the curve g_{P_n} separates k from h and from l. For suppose there are two such integers n_1 and n_2. Then one of the curves $g_{P_{n_1}}$ and $g_{P_{n_2}}$ separates the other one from h and therefore separates a point of β from h. But every domain that contains a point of β contains a point of some G-curve that separates h from k and from l. Hence either $g_{P_{n_1}}$ or $g_{P_{n_2}}$ separates h

from a G-curve g_0 which separates h from k and from l. Hence $g_{P_{n_1}}$ or $g_{P_{n_2}}$ has at least one point in common with g_0. But this is contrary to hypothesis. Similarly there does not exist more than one value of n such that g_{P_n} separates l from h and from k. It follows that there exists an infinite sub-sequence $g_{P_{n_1}}, g_{P_{n_2}}, g_{P_{n_3}}, \cdots$ of distinct curves of the sequence $g_{P_1}, g_{P_2}, g_{P_3}, \cdots$ and an arc XY (identical with one of the arcs $\overline{AB}$, $\overline{CD}$ and $\overline{EF}$) such that for each m the curve $g_{P_{n_m}}$ contains an interval $A_{P_{n_m}} P_{n_m} B_{P_{n_m}}$ whose endpoints A_{P_n} and B_{P_n} are on XY and which lies, except for its endpoints, wholly within $\bar{J}$. By hypothesis, for every positive number ϵ there exists a positive number $\delta_{\bar{J}\epsilon}$ such that if, for some n, the distance from $A_{P_{n_m}}$ to $B_{P_{n_m}}$ is less than $\delta_{\bar{J}\epsilon}$ then the whole arc $A_{P_{n_m}} P_{n_m} B_{P_{n_m}}$ lies within a circle of radius ϵ. It can be easily seen that if i and j are distinct integers the intervals $A_{P_{n_i}} B_{P_{n_i}}$ and $A_{P_{n_j}} B_{P_{n_j}}$ of the arc XY have no point in common. Hence if ϵ is the least distance from a point of the arc XY to a point of the closed curve J there exists an integer $\bar{m}$ such that the distance from $A_{P_{n_{\bar{m}}}}$ to $B_{P_{n_{\bar{m}}}}$ is less than $\delta_{\bar{J}\epsilon}$. It follows that every point of the arc $A_{P_{n_{\bar{m}}}} P_{n_{\bar{m}}} B_{P_{n_{\bar{m}}}}$ is at a distance of less than ϵ from the point $A_{P_{n_{\bar{m}}}}$. But the distance from $P_{n_{\bar{m}}}$ to $A_{P_{n_{\bar{m}}}}$ is not less than ϵ. Thus the supposition that there exists no last curve of class I with respect to h has led to a contradiction. Hence there exists a curve $\bar{h}$ which is the last curve of class I with respect to h. In a similar way it may be shown that there exist curves $\bar{k}$ and $\bar{l}$ which are the last curves of class I with respect to k and l respectively. No one of the curves $\bar{h}$, $\bar{k}$ and $\bar{l}$ separates the other two from each other and no curve of the set G separates one of them from the other two.

Let $\bar{D}$ denote the connected domain which is bounded by the curves $\bar{h}$, $\bar{k}$ and $\bar{l}$. With the aid of several applications of the fact that the system G is inversely equicontinuous with respect to every bounded set of points it can be shown that there exist six points $\bar{M}$, $\bar{T}$, $\bar{H}$, $\bar{L}$, $\bar{K}$, $\bar{N}$ and three arcs $\bar{M}\bar{N}$, $\bar{T}\bar{H}$ and $\bar{K}\bar{L}$ such that (1) $\bar{M}$ and $\bar{T}$ are on $\bar{h}$, $\bar{H}$ and $\bar{L}$ are on $\bar{l}$, $\bar{K}$ and $\bar{N}$ are on $\bar{k}$, (2) $\bar{M}\bar{N}$, $\bar{T}\bar{H}$ and $\bar{K}\bar{L}$ lie, except for their endpoints, entirely in $\bar{D}$ and no two of them have a point in common, (3) no curve of G distinct from $\bar{h}$, $\bar{k}$ and $\bar{l}$ contains a point of more than one of the arcs $\bar{M}\bar{N}$, $\bar{T}\bar{H}$ and $\bar{K}\bar{L}$ (Fig. 6). Let $\bar{\beta}$ denote the region bounded by the arcs $\bar{M}\bar{N}$, $\bar{T}\bar{H}$, $\bar{K}\bar{L}$ and the intervals $\bar{M}\bar{T}$, $\bar{H}\bar{L}$ and $\bar{K}\bar{N}$ respectively of the curves $\bar{h}$, $\bar{l}$ and $\bar{k}$. By an argument similar in large part to that employed above to show the existence of $\bar{h}$ it may be proved that if g is a curve of the set G that contains a point of $\bar{M}\bar{N}$ there exists a curve $\bar{g}$ of the set G which either coincides with g or separates g from each of the curves $\bar{h}$, $\bar{k}$ and $\bar{l}$ but is not itself separated from any one of these curves by any other curve of G. Every such curve $\bar{g}$ will be called a curve of class III. There clearly exist infinitely many distinct curves of class III. Let M^* denote a point on $\bar{h}$ in the order $\bar{T}\bar{M}M^*$ and let N^* denote a

point on $\bar{k}$ in the order $\bar{K}\bar{N}N^*$. Let t denote an arc that has M^* and N^* as endpoints, lies except for its endpoints entirely in $\bar{D}$ and has no point in common with $\bar{M}\bar{N}$. If a curve of G contains a point P of $\bar{M}\bar{N}$ there is an interval of that curve that contains P and has its endpoints on t. It follows that there exists an infinite set of distinct arcs $A_1P_1B_1$, $A_2P_2B_2$, $A_3P_3B_3$, $\cdots$ such that (1) for every n, A_n and B_n are on t and the arc $A_nP_nB_n$ is an interval of a curve of class III, (2) if n_1 is distinct from n_2, $A_{n_1}P_{n_1}B_{n_1}$ and $A_{n_2}P_{n_2}B_{n_2}$

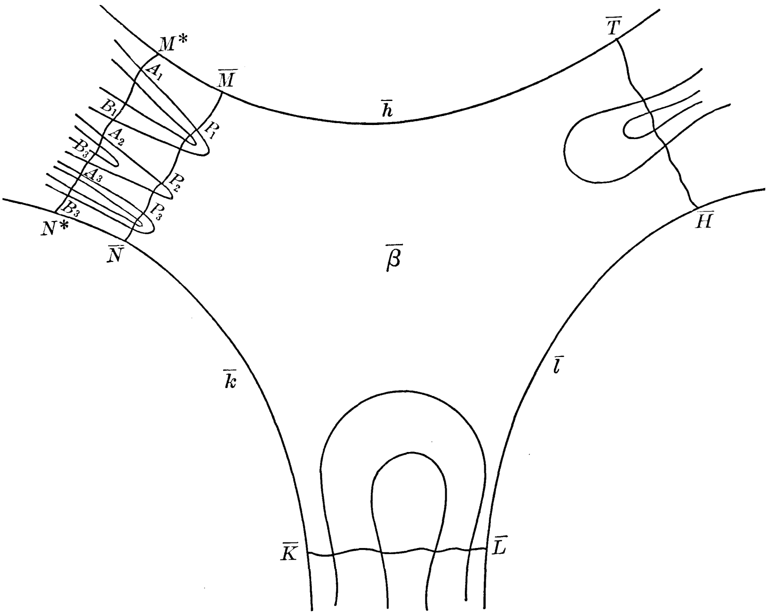

FIG. 6.

are not intervals of the same curve of the set G, (3) for every n the arc $A_nP_nB_n$ contains a point of $\bar{M}\bar{N}$, (4) if n_1 is distinct from n_2 the intervals $A_{n_1}B_{n_1}$ and $A_{n_2}B_{n_2}$ of M^*N^* have no point in common. If ϵ is the least distance from a point of t to a point of $\bar{M}\bar{N}$ there exists a positive integer n such that the distance from $A_{\bar{n}}$ to $B_{\bar{n}}$ is less than $\delta_{t\epsilon}$. But there exists an interval $A_{\bar{n}}P_{\bar{n}}B_{\bar{n}}$ of a curve of the set G with $A_{\bar{n}}$ and $B_{\bar{n}}$ as endpoints and containing a point $P_{\bar{n}}$ at a distance of ϵ or more from $A_{\bar{n}}$. Thus the supposition that no one of the curves h, k and l separates the other two has led to a contradiction. It follows that, of any three curves of G, one separates the other two.

If $\bar{g}$ is a definite G-curve, C is a definite circle and A is a point of C not lying on $\bar{g}$ there exists a G-curve which lies on the A-side of $\bar{g}$ but contains

* See Theorem 2 of my paper *On the most general class L of Fréchet in which the Heine-Borel-Lebesgue Theorem holds true*, Proceedings of the National Academy of Sciences, vol. 5 (1919), p. 208.

no point of C. This may be proved as follows. For every G-curve g that lies on the A-side of $\bar{g}$ and contains a point of C let C_g denote the set of all points $[X]$ of C such that X is either on g or on the far side of g from $\bar{g}$. For every such g the set C_g is closed and bounded and for every two such g's, $\bar{g}_1$ and $\bar{g}_2$, either $C_{\bar{g}_1}$ contains $C_{\bar{g}_2}$ or $C_{\bar{g}_2}$ contains $C_{\bar{g}_1}$. It follows* that there exists at least one point P which belongs to C_g for every G-curve g which lies on the A-side of $\bar{g}$ and contains a point of C. If g_P denotes that G-curve which contains the point P then no G-curve which lies on the far side of g_P from $\bar{g}$ can contain any point of C.

It easily follows that for every circle C there exist two G-curves such that every point of C lies between them. Now let O denote some definite point and for each positive integer n let C_n denote a circle with center at O and radius n. Let g_0 denote that G-curve which passes through O. Let g_1 and g_{-1} denote two G-curves such that C_1 lies between them. Let g_2 and g_{-2} denote two G-curves such that C_2 lies between them and such that g_2 is on the far side of g_1 from O and g_{-2} is on the far side of g_{-1} from O. This process may be continued. It follows that there exists a set G_0 of G-curves consisting of two infinite sequences $g_0, g_1, g_2, \cdots$ and $g_{-1}, g_{-2}, g_{-3}, \cdots$ such that, for each positive n, C_n lies between g_n and g_{-n}, g_{n+1} is on the far side of g_n from O and $g_{-(n+1)}$ is on the far side of g_{-n} from O. It is clear that every point is either on some curve of the set G_0 or between two successive curves of G_0. With the use of the fact that the system G is inversely equicontinuous with respect to every bounded point-set and that, of any three curves of G, one separates the other two, it can be shown that there exist four infinite sequences of points $A_0, A_1, A_2, A_3, \cdots$; $A_{-1}, A_{-2}, A_{-3}, \cdots$; $B_0, B_1, B_2, B_3, \cdots$ and $B_{-1}, B_{-2}, B_{-3}, \cdots$ and two sequences of arcs $A_0 B_0$, $A_1 B_1$, $A_2 B_2, \cdots$ and $A_{-1} B_{-1}$, $A_{-2} B_{-2}$, $A_{-3} B_{-3}, \cdots$ such that (1) for every n the points A_n, A_{n+1}, A_{n+2} are in the order $A_n A_{n+1} A_{n+2}$ on g_1 and the points B_n, B_{n+1}, B_{n+2} are in the order $B_n B_{n+1} B_{n+2}$ on g_0, (2) for each n and $m\,(m \neq n)$ the arcs $A_n B_n$ and $A_m B_m$ lie entirely between g_0 and g_1 and have no point in common, (3) for every point X on g_1 and every point Y on g_0 there exists a positive integer n such that X is on the interval $A_{-n} A_n$ of g_1 and Y is on the interval $B_{-n} B_n$ of g_0, (4) if, for each n, J_n denotes the closed curve formed by the arcs $A_n B_n$, $A_{n+1} B_{n+1}$ and the G-intervals $A_n A_{n+1}$ and $B_n B_{n+1}$ then (a) every point between g_0 and g_1 is on or within some J_n, (b) if $|m - n| > 1$ every G-interval whose endpoints are on or within J_m lies wholly without J_n. For each integer n let K_n denote the set of all points $[X]$ such that X lies on a G-interval whose endpoints are within J_{4n}. By methods wholly or largely identical with those employed in the proof of Lemma 1 it may be shown that there exists an arc $D_n E_n$ which lies entirely in the domain K_n, except that its endpoints D_n and E_n lie on g_1 and g_0 respectively,

and which does not have more than one point in common with any curve of the set G. For each n let $\bar{J}_n$ denote the closed curve found by the arcs $D_n E_n$, $D_{n+1} E_{n+1}$ and the G-intervals $D_n D_{n+1}$, $E_n E_{n+1}$ and let $\bar{R}_n$ denote its interior. By Theorem 1 there exists a set of arcs α_n such that (1) each arc of α_n has its endpoints on g_1 and g_0 respectively and lies, except for its endpoints, wholly within $\bar{J}_n$, (2) no two arcs of α_n have a point in common, (3) through each point of the point-set composed of $\bar{R}_n$ and the two G-segments $D_n D_{n+1}$ and $E_n E_{n+1}$ there is one and only one arc of the set α_n, (4) no arc of α_n has more than one point in common with any one arc of the set G. Let H_0 denote the set of arcs composed of all the arcs of all the sets α_n together with all the arcs $D_n E_n$. For each n there exists a set of arcs H_n bearing to g_n and g_{n+1} a relation similar to the above described relation of H_0 to g_0 and g_1, so that (1) each arc of H_n lies entirely between g_n and g_{n+1} except that its endpoints are on g_n and g_{n+1} respectively, (2) through each point that lies on g_n or g_{n+1} or between them there is just one arc of H_n, (3) no arc of H_n has more than one point in common with any arc of the set G. For each point P there exists n_P such that P is either on g_{n_P} or between g_{n_P} and g_{n_P+1}. Let h_{1P} denote that arc of H_{n_P} which passes through P. Let h_{2P} denote that arc of H_{n_P+1} which has an endpoint in common with h_{1P} and let h_{0P} denote that arc of H_{n_P-1} which has an endpoint in common with h_{1P}. This process may be continued. Thus there exists a set of arcs $[h_{mP}]$ $(-\infty < m < \infty)$ such that, for every m, $h_{(m+1)P}$ belongs to the set H_{n_P+m} and has an endpoint in common with $h_{(m+2)P}$. The point-set obtained by adding together all the arcs of the set $[h_{mP}]$ is an open curve h_P that passes through the point P and has just one point in common with each curve of the set G. Let H denote the set of all curves h_P for all points P of S. Through each point of S there is just one curve of the set H and just one curve of the set G and if h is any curve of H and g is any curve of G, h and g have just one point in common. It follows* that there exists a one to one transformation of S into itself which carries H into a complete system of parallel lines and G into another complete system of parallel lines.

THEOREM 3. *If $A\bar{A}_0 \bar{B}_0 B$ is a rectangle and $A_1 B_1$, $A_2 B_2$, $A_3 B_3$, $\cdots$ is an infinite sequence G of arcs such that* (1) *the points A_1, A_2, A_3, $\cdots$ are in the order $\bar{A}_0 A_1 A_2 A_3 \cdots A_n A_{n+1} \cdots A$ on the interval $\bar{A}_0 A$ and the points B_1, B_2, B_3, $\cdots$ are in the order $\bar{B}_0 B_1 B_2 B_3 \cdots B_n B_{n+1} \cdots B$ on the interval $\bar{B}_0 B$,* (2) *every arc of G lies except for its endpoints entirely within the rectangle $A\bar{A}_0 \bar{B}_0 B$,* (3) *no two arcs of G have a point in common and* (4) *for each positive number ϵ there exists a positive number n_ϵ such that if $n > n_\epsilon$, then every point of $A_n B_n$ is at a distance less than ϵ from the line AB; then in order that the sequence G*

* Cf. pp. 177–178 of my paper *Concerning a set of postulates for plane analysis situs*, loc. cit.

should be equivalent from the standpoint of analysis situs to an infinite sequence of straight line intervals, satisfying the same conditions (1)–(4), *and all parallel to* AB, *it is necessary and sufficient that the set of arcs* G *should be equicontinuous.*

That this condition is necessary, is evident. I will show that it is sufficient.

Proof. Suppose G is an equicontinuous sequence of arcs satisfying conditions (1)–(4) of the hypothesis of Theorem 3. By hypothesis for every positive number ϵ there exists a positive number δ_ϵ such that if P_1 and P_2 are two points on an arc g of G at a distance apart less than or equal to δ_ϵ then the interval $P_1 P_2$ of g lies entirely within some circle of radius ϵ. It follows with the help of condition (4) that if X and Y are two points of AB at a distance apart less than or equal to δ_ϵ and p_1 and p_2 are straight lines perpendicular to AB at X and Y respectively then if* $n > n_{\delta_\epsilon}$ no interval of $A_n B_n$ with endpoints on p_1 contains a point of p_2. If, for every n, X_n denotes the last point that $A_n B_n$ has in common with p_1 and Y_n denotes the first point that it has in common with p_2 it follows that if n_1 and n_2 are positive integers greater than n_{δ_ϵ} and P_{n_1} and P_{n_2} are points between p_1 and p_2 on the intervals $X_{n_1} Y_{n_1}$ and $X_{n_2} Y_{n_2}$ respectively of the arcs $A_{n_1} B_{n_1}$, $A_{n_2} B_{n_2}$ then P_{n_1} can be joined to P_{n_2} by a simple continuous arc that lies wholly between p_1 and p_2 and lies except for its endpoints wholly between the arcs $A_{n_1} B_{n_1}$ and $A_{n_2} B_{n_2}$. Now for each positive integer n subdivide the interval AB into 3^n equal subintervals by $3^n - 1$ points $A_{n1}, A_{n2}, A_{n3}, \cdots, A_{n(3^n-1)}$ $(1 \leqq n < \infty)$ in the order $AA_{n_1} A_{n_2} \cdots A_{n(3^n-1)} B$. For each n and m $(1 \leqq m \leqq 3^n - 1)$ let p_{nm} denote the perpendicular to AB at the point A_{nm}. There exists a sequence of positive integers $\bar{n}_1, \bar{n}_2, \bar{n}_3, \cdots$ such that $\bar{n}_1 < \bar{n}_2 < \bar{n}_3 \cdots$ and such that, for every k, $\bar{n}_k > n_{\delta_{l/3^k}}$, where l is the length of AB. For each k let $\bar{g}_k$ denote the arc $A_{\bar{n}_k} B_{\bar{n}_k}$ and let $\bar{A}_k$ and $\bar{B}_k$ denote its endpoints, $\bar{A}_k$ being that one which lies on $A\bar{A}_0$. For each n and m $(1 \leqq m \leqq 3^n - 1)$ let B_{nm} be the first and A_{nm} the last point that the arc $\bar{g}_n$ has in common with p_{nm}. Let t_{nm} denote the G-segment $A_{nm} B_{n(m+1)}$. For each n and each positive integer m (less than 3^n) of the form $3k - 2$ (where k is an integer) let X_{nm} denote a point of the segment t_{nm}. If, for each such n and m, $\bar{m}$ denotes the number $3m + 1$, there exists (Fig. 7) an arc $X_{nm} X_{(n+1)\bar{m}}$ which has not† more than one point in common with any arc of the set G, lies wholly between the lines p_{nm} and $p_{n(m+1)}$ and also lies, except for its endpoints, wholly between the arcs $\bar{g}_n$ and $\bar{g}_{n+1}$. For every n $(0 \leqq n < \infty)$ let A_n' denote a point on the straight line interval $\bar{A}_0 A$ at a distance from A equal to $a/(n+1)$, where a is the length of $\bar{A}_0 A$, and let B_n' denote a point on the interval $\bar{B}_0 B$ at the distance $a/(n+1)$ from B. Let $\bar{g}_0$ denote the straight line interval $\bar{A}_0 \bar{B}_0$. For each n $(0 \leqq n < \infty)$ let J_n denote the closed curve formed by the arcs $\bar{g}_n$ and $\bar{g}_{n+1}$ and the intervals

* For the meaning of n_{δ_ϵ} see Condition (4).

† There are not more than a finite number of arcs of the set G between $\bar{g}_n$ and $\bar{g}_{n+1}$.

$\overline{A}_n\,\overline{A}_{n+1}$ and $\overline{B}_n\,\overline{B}_{n+1}$ of $A\overline{A}_0$ and $B\overline{B}_0$ respectively. Let R_n denote the point-set composed of J_n and its interior. Let R'_n denote the point-set composed of the rectangle $A'_n\,A'_{n+1}\,B'_{n+1}\,B'_n$ and its interior and let R be that composed of $A\overline{A}_0\,\overline{B}_0\,B$ and its interior. Let b denote the length of AB. With the aid of a theorem of Schoenflies'* it may be easily seen that there exists a sequence of one to one transformations $T_0,\ T_1,\ T_2,\ \cdots$ such that, for each n ($0 \leqq n < \infty$), (1) T_n is a continuous transformation of R_n into R'_n, (2) T_n trans-

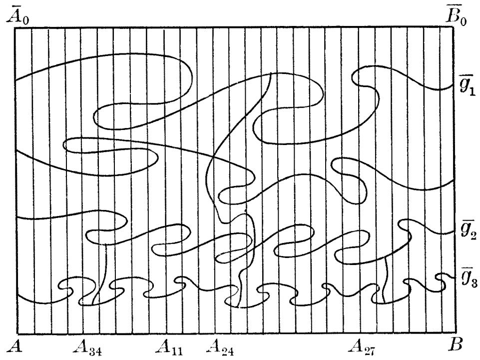

FIG. 7.

forms $\overline{A}_n$, $\overline{A}_{n+1}$, $\overline{B}_n$ and $\overline{B}_{n+1}$ into A'_n, A'_{n+1}, B'_n and B'_{n+1} respectively, (3) if P is a point of $\overline{g}_{n+1}$, $T_n(P) = T_{n+1}(P)$, (4) if there is any G-arc between $\overline{g}_n$ and $\overline{g}_{n+1}$ every such arc is transformed by T_n into a straight line interval parallel to AB, (5) if $n \geqq 1$ then for each m (less than 3^n) of the form $3k - 2$, where k is a positive integer, the point X_{nm} is transformed by T_n into a point X'_{nm} lying on the straight interval $A'_n\,B'_n$ at a distance from $A\overline{A}_0$ equal to $(m + 1/2)\,b/3^n$ and the arc $X_{nm}\,X_{(n+1)\overline{m}}$ is transformed into the straight line interval joining the point X'_{nm} to the point $X'_{(n+1)\overline{m}}$. For each n let H_n denote the set of all arcs $[h]$ in R_n such that $T_n(h)$ is a vertical† straight line interval. If $\overline{P}$ is a point on $\overline{A}_0\,\overline{B}_0$ let $h_{\overline{P}0}$ denote that arc of H_0 which contains $\overline{P}$, let $h_{\overline{P}1}$ denote that arc of H_1 which has an endpoint in common with $h_{\overline{P}0}$, let $h_{\overline{P}2}$ denote that arc of H_2 which has an endpoint in common with $h_{\overline{P}1}$, and so on indefinitely. It is possible to show that there exists only one point $O_{\overline{P}}$ on AB which is a limit point of the point-set $h_{\overline{P}0} + h_{\overline{P}1} + h_{\overline{P}2} + \cdots$ and that the set of points $O_{\overline{P}} + h_{\overline{P}0} + h_{\overline{P}1} + h_{\overline{P}2} + \cdots$ is a simple continuous arc from $\overline{P}$ to $O_{\overline{P}}$. Let H denote the set of all such arcs for all points $\overline{P}$ on $\overline{A}_0\,\overline{B}_0$. Let K denote the set of arcs composed of AB and every arc in R which, for some n, is transformed by T_n into a straight interval parallel to AB. If P is any point of R let O_P denote the point which AB has in common with

* *Bericht über die Entwickelung der Lehre von den Punktmannigfaltigkeiten,* Part II, p. 108.

† I.e., perpendicular to AB.

that arc of H which passes through P and let L_P denote the point which $\overline{A_0 A}$ has in common with that arc of K which passes through P. For each point P, of R, let $T(P)$ denote the point in which the perpendicular to AB at the point O_P intersects the perpendicular to $\overline{AA_0}$ at the point L_P. The so determined transformation T is a continuous transformation of R into itself. It is easy to see that there exists a continuous transformation of S into itself which reduces to T on R. Every such transformation satisfies the requirements of Theorem 3.

The truth of the following theorems may also be established.

THEOREM 4. *If, in a plane S, O is a point and G is a set of open curves through O such that through each point of S distinct from O there is one and only one curve of the set G, then in order that G should be equivalent from the standpoint of analysis situs to the set of all straight lines in S through O it is necessary and sufficient that G should be equicontinuous with respect to every bounded set of points.*

THEOREM 5. *If, in a plane S, O is a point and G is a set of simple closed curves enclosing O such that through each point of S distinct from O there is one and only one curve of the set G, then in order that G should be equivalent from the standpoint of analysis situs to the set of all circles in S with center at O it is necessary and sufficient that the set G should be equicontinuous with respect to every bounded set of points.*

UNIVERSITY OF TEXAS